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Author(s)	KATO, Mikio; TAKAHASHI, Yasuji
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Some characterizations of uniformly non-square Banach spaces

Mikio KATO (加藤幹雄) and Yasuji TAKAHASHI (高橋泰嗣)

Kyushu Inst. Tech. and Okayama Prefect. Univ.

In this note we present a sequence of characterizations of uniformly non-square Banach spaces which was recently given by the authors [15]. Some of them are similar to the well-known homogeneous characterization of uniformly convex spaces. As direct consequences we have the following: (i) A Banach space X is uniformly non-square if and only if the von Neumann-Jordan (NJ-) constant for X is less than 2. (Note that X is super-reflexive if and only if X admits an equivalent norm with NJ-constant less than 2.) (ii) Uniform non-squareness is inherited by dual spaces; this seems not to have appeared in literature. (iii) $L_p(X)$ ($1 < p < \infty$) is uniformly non-square if and only if X is (Smith and Turett [14]).

Let X be a Banach space. Let B_X denote the closed unit ball of X. X is called *uniformly convex* if for any $\varepsilon > 0$ ($0 < \varepsilon < 2$) there exists a $\delta > 0$ such that $\|(x + y)/2\| < 1 - \delta$, whenever $\|x - y\| \geqq \varepsilon$, $x, y \in B_X$. X is called *uniformly non-square* ([7]) if there exists a $\delta > 0$ such that $\|(x+y)/2\| \leqq 1 - \delta$, whenever $\|(x-y)/2\| > 1 - \delta$, $x, y \in B_X$. The *von Nuemann-Jordan (NJ-) constant* for X ([4]), we denote it by $C_{NJ}(X)$, is the smallest constant for which

$$\frac{1}{C} \leqq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leqq C \qquad \forall (x, y) \neq (0, 0).$$

We refer the reader to [9], [11], [12], [15] for some recent results on NJ-constant.)

It is well known that uniformly convex spaces are uniformly non-square, and uniformly non-square spaces are super-reflexive, or equivalently uniformly convexifiable (cf. [1]).

Now, Recall the following well-known homogeneous characterization of uniformly convex spaces:

A. Proposition (cf. [1]). Let $1 < p < \infty$. A Banach space X is uniformly convex if and only if for any $\varepsilon > 0$ there exists $\delta = \delta_p(\varepsilon) > 0$ such that $\|x - y\| \geqq 2(1 - \varepsilon)$, $x, y \in B_X$ implies

$$\left\|\frac{x+y}{2}\right\|^p \leqq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{1}$$

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Let $l_r^2(X)$ denote the X-valued l_r^2-space.

1. Theorem (Takahashi and Kato [15]). Let $1 < p < \infty$. For a Banach space X the following are equivalent:

(i) X; uniformly non-square.

(ii) There exist ε and δ $(0 < \varepsilon, \delta < 1)$ such that if $\|x - y\| \geqq 2(1 - \varepsilon)$, $x, y \in B_X$, then

$$\left\|\frac{x+y}{2}\right\|^p \leqq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{2}$$

(iii) There exists a δ $(0 < \delta < 1)$ such that if $\|x - y\| \geqq 2(1 - \delta)$, $x, y \in B_X$, then

$$\left\|\frac{x+y}{2}\right\|^p \leqq (1-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{3}$$

(iv) There exists a δ $(0 < \delta < 2)$ such that for any $x, y \in X$,

$$\left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p \leqq (2-\delta)\frac{\|x\|^p + \|y\|^p}{2}. \tag{4}$$

(v) $\| A : l_p^2(X) \to l_p^2(X) \| < 2$.

(vi) For any (resp. some) $1 < r \leqq \infty$, $1 \leqq s < \infty$,

$$\| A : l_r^2(X) \to l_s^2(X) \| < 2^{1/r' + 1/s},$$

where $1/r + 1/r' = 1$.

(vii) $C_{NJ}(X) < 2$.

2. **Remarks.** (i) For the case $p = 1$, the homogeneous characterization of uniform convexity stated in Proposition A fails to hold (put $y = 0$), whereas the corresponding characterizations of uniform non-squareness (ii) and (iii) given in Theorem 1 remain valid; the assertions (iv) and (v) are false for $p = 1$.

(ii) For any Banach space X and for any $1 \leqq p \leqq \infty$ it holds that

$$\| A : l_p^2(X) \to l_p^2(X) \| \leqq 2.$$

For $X = L_p$ we have

$$\| A : l_p^2(L_p) \to l_p^2(L_p) \| = 2^{1/\min(p, p')},$$

which is equivalent to the following Clarkson's inequality:

$$(\| f+g \|_p^p + \| f-g \|_p^p)^{1/p} \leqq 2^{1/\min(p, p')} (\| f \|_p^p + \| g \|_p^p)^{1/p} \quad (\forall f, \ g \in L_p)$$

where equality is attained (Clarkson [3]).

(iii) Smith and Turett [14; esp. Lemma 14] gave a characterization of uniformly non-$l_1(n)$ Banach spaces, which in particular implies the equivalence of (i) and (iv) of Theorem 1 as the case $n = 2$ (their proof differs from ours in [15]). On the other hand, J. J. Schäffer (cf. [13], esp. p. 131) introduced a different notion of uniform non-squareness, which is known to be equivalent to James' treated here.

(iv) For any Banach space X

$$\| A : l_r^2(X) \to l_s^2(X) \| \leqq 2^{1/r'+1/s} \quad \text{for all } 1 \leqq r,\ s \leqq \infty$$

and

$$\| A : l_r^2(X) \to l_s^2(X) \| = 2^{1/r'+1/s} \quad \text{for } r = 1 \text{ or } s = \infty.$$

For $X = L_p$ $(1 \leqq p \leqq \infty)$ we have

$$(5) \quad \| A : l_r^2(L_p) \to l_s^2(L_p) \| = 2^{c(r,s;p)} \quad \text{for all } 1 \leqq r,\ s \leqq \infty,$$

where $c(r,s;p) = \max\{1/r',\ 1/s,\ 1/r'+1/s-1/\max(p,p')\}$, which yields the following Clarkson-Boas-Koskela's inequality:

$$(6) \quad (\| f+g \|_p^s + \| f-g \|_p^s)^{1/s} \leqq 2^{c(r,s;p)} (\| f \|_p^r + \| g \|_p^r)^{1/r} \quad \text{for } \forall f,\ g \in L_p$$

(see [10]).

As Boas [2] observed (cf. Clarkson [3]), the inequality (6), or (5), with $c(r,s;p) = 1/r'$ implies the uniform convexity of L_p. The same is clearly true for a general Banach space X; that is, if

$$(7) \qquad \| A : l_r^2(X) \to l_s^2(X) \| \leqq 2^{1/r'}$$

with some $1 \leqq r,\ s < \infty$, then X is uniformly convex. (Note that if (7) is valid, then (7) is in fact reduced to identity; $(x,\ x)$, $x \neq 0$, is norm-attaining.) As is seen in the example below, this fails to be valid if the above norm of A is greater than $2^{1/r'}$.

Theorem 1 enables us to understand difference between uniform convexity and uniform non-squareness via behavior of norms of the Littlewood matrix:

3. **Example.** Let $1 < p \leqq 2$ and $1 < \lambda < 2^{1/p'}$. Let $X_{p,\lambda}$ be the space $l_{p'}$ equipped with the norm $\|x\|_{p,\lambda} := \max\{\|x\|_{p'}, \lambda\|x\|_\infty\}$, where $1/p + 1/p' = 1$. Then, in the same way as the proof of Proposition 1 in [15] we have for $p \leqq r < \infty$

$$(8)\quad 2^{1/r'} < \|A: l_r^2(X_{p,\lambda}) \to l_{p'}^2(X_{p,\lambda})\| = \lambda 2^{1/r'} < 2^{1/r'+1/p'}.$$

By Proposition 1 of [15], $X_{p,\lambda}$ is not uniformly convex (nor strictly convex) for all $1 < \lambda < 2^{1/p'}$, whereas Theorem 1 asserts that $X_{p,\lambda}$ is uniformly non-square (compare (8) with (7)).

Now, Theorem 1 immediately yields the following results.

4. **Corollary.** (i) The dual space X' of X is uniformly non-square if and only if X is.

(ii) Let $1 < p < \infty$. Then, the Lebesgue-Bochner space $L_p(X)$ is uniformly non-square if and only if X is (Smith and Turett [14]; see also [15]).

Indeed the assertion (i) is a direct consequence of Theorem 1 (vi) since $C_{NJ}(X') = C_{NJ}(X)$, which is easily seen (cf. [12; Proposition B]). For the if-part of (ii) integrate the inequality (4) and use Theorem 1 (iv) (see [14]).

5. **Remark.** The above result (i) of Corollary 4 seems not to have appeared in literature. Note here that uniform convexity is not inherited by the dual (cf. [1]). Giesy [5] showed that the bidual X'' is uniformly non-$l_1(n)$ if and only if X is (this is evident for $n = 2$, namely for uniform non-squareness, since uniformly non-square spaces are reflexive; James [7]); and the dual space X' is B-convex (uniformly non-$l_1(n)$ for

some n) if and only if X is. It is known that for some Orlicz spaces uniform non-squareness coincides with reflexivity and also with B-convexity ([6]).

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